

REPRESENTATION THEORY OF THE INFINITE SYMMETRIC GROUP AND PFAFFIAN POINT PROCESSES

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ABSTRACT. We construct a family of Pfaffian point processes relevant for the harmonic analysis on the infinite symmetric group. The correlation functions of these processes are representable as Pfaffians with matrix valued kernels. We give explicit formulae for the matrix valued kernels in terms of the classical Whittaker functions. The obtained formulae have the same structure as that arising in the study of symplectic ensembles of Random Matrix Theory.

The paper is an extended version of the author's talk at Fall 2010 MSRI Random Matrix Theory program.

1. INTRODUCTION

Let $S(\infty)$ denote the group whose elements are finite permutations of $\{1, 2, \dots\}$. The group $S(\infty)$ is called the infinite symmetric group, and it is a model example of a “big” group. The harmonic analysis for such groups is an active topic of modern research, with connections to different areas of mathematics from enumerative combinatorics to random growth models and to the theory of Painlevé equations. A theory of harmonic analysis on the infinite symmetric and infinite-dimensional unitary groups is developed by Kerov, Olshanski and Vershik [13, 14], Borodin and Olshanski [5, 3, 8, 6]. For an introduction to harmonic analysis on the infinite symmetric group see Olshanski [17]. A paper by Borodin and Deift [4] studies differential equations arising in the context of harmonic analysis on the infinite-dimensional unitary group, and a paper by Borodin and Olshanski [7] describes the link to problems of enumerative combinatorics, and to certain random growth models.

Borodin and Olshanski [5, 3, 8, 6] have shown that the problem of the harmonic analysis on the infinite symmetric group leads to determinantal point processes, which in many ways are similar to point processes associated with random matrix ensembles of the $\beta = 2$ symmetry class. On the other hand, in addition to ensembles of $\beta = 2$ symmetry class, Random Matrix Theory deals with ensembles of $\beta = 1$ and $\beta = 4$ symmetry classes. These ensembles (in contrast to those from $\beta = 2$ symmetry class) lead to Pfaffian point processes. In addition to random matrix problems, Pfaffian point processes appear in the theory of random partitions (see

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Rains [20], Borodin and Rains [9], Forrester, Nagao and Rains [11], Strahov [21, 23], Petrov [19]), and in statistical mechanics (see, for example, Ferrari [10]).

In this paper we construct and investigate Pfaffian point processes relevant for the harmonic analysis on the infinite symmetric group. In particular, we present explicit formulae for the correlation functions, see Theorem 4.1.

This paper is organized as follows. Section 2 reviews the construction of a remarkable family of unitary representations $T_{z, \frac{1}{2}}$ associated with the infinite-dimensional pair $(S(2\infty), H(\infty))$ ($(S(2\infty), H(\infty))$ is a Gelfand pair in the sense of Olshanski [16]). Section 3 gives a spectral representation of the spherical functions of $T_{z, \frac{1}{2}}$, and introduces the spectral measures $M_{z, \theta=\frac{1}{2}}^{\text{Spectral}}$ (these objects are closely related with the z -measures with the Jack parameter $\theta = \frac{1}{2}$). The spectral measures $M_{z, \theta=\frac{1}{2}}^{\text{Spectral}}$ govern decomposition of $T_{z, \frac{1}{2}}$ into irreducible components, and the irreducible components are parameterized by points of the Thoma set.

The problem of harmonic analysis under considerations is to describe $M_{z, \theta=\frac{1}{2}}^{\text{Spectral}}$. This is done in Section 4. We use the idea proposed by Borodin and Olshanski and convert the points of the Thoma set into a point configuration. Then the spectral measure $M_{z, \theta=\frac{1}{2}}^{\text{Spectral}}$ defines a random point processes which can be studied by standard tools of probability theory. Theorem 4.1 is the main result of the present paper. It states that the point process defined by the (lifted) spectral measure $\widetilde{M}_{z, \theta=\frac{1}{2}}^{\text{Spectral}}$ is a Pfaffian point processes. Moreover, Theorem 4.1 gives an explicit formula for the corresponding correlation functions in terms of the classical Whittaker functions. It is remarkable that the correlation kernel of Theorem 4.1 has the same structure as the matrix Airy kernel arising in the study of the symplectic random matrix ensembles. This similarity suggests a possibility to study the obtained Pfaffian point process on the same level as Pfaffian point processes of Random Matrix Theory.

Finally, Section 5 contains the sketch of the proof of Theorem 4.1.

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2. THE REPRESENTATIONS $T_{z, \frac{1}{2}}$

In this section we review the construction of family $T_{z, \frac{1}{2}}$ of unitary representations of the group $S(2\infty)$. These representations (introduced in Olshanski [18], Strahov [22]) are parameterized by points $z \in \mathbb{C} \setminus \{0\}$, and can be viewed as analogues of the generalized regular representations introduced in Kerov, Olshanski, and Vershik [13, 14].

2.1. The spaces $X(n)$ and their projective limit. Let $S(2n)$ be the permutation group of $2n$ symbols realized as the group of permutations of the set $\{-n, \dots, -1, 1, \dots, n\}$. Let $t \in S(2n)$ be the product of the transpositions $(-n, n), (-n+1, n-1), \dots, (-1, 1)$.

By definition, the group $H(n)$ is the centralizer of \check{t} in $S(2n)$,

$$H(n) = \left\{ \sigma \mid \sigma \in S(2n), \sigma \check{t} \sigma^{-1} = \check{t} \right\}.$$

The group $H(n)$ is called the hyperoctahedral group of degree n . It is known that $(S(2n), H(n))$ is a Gelfand pair, see Macdonald [15], VII, §2.

Set $X(n) = H(n) \backslash S(2n)$, so $X(n)$ is the space of right cosets of the subgroup $H(n)$ in $S(2n)$. The set $X(n)$ can be realized as the set of all pairings of $\{-n, \dots, -1, 1, \dots, n\}$ into n unordered pairs. Thus every element \check{x} of $X(n)$ may be represented as a collection of n unordered pairs,

$$(2.1) \quad \check{x} \in X(n) \longleftrightarrow \check{x} = \left\{ \{i_1, i_2\}, \dots, \{i_{2n-1}, i_{2n}\} \right\},$$

where i_1, i_2, \dots, i_{2n} are distinct elements of the set $\{-n, \dots, -1, 1, \dots, n\}$.

Given an element $\check{x}' \in X(n+1)$ we define its derivative element $\check{x} \in X(n)$ as follows. Represent \check{x}' as $n+1$ unordered pairs. If $n+1$ and $-n-1$ are in the same pair, then \check{x} is obtained from \check{x}' by deleting this pair. Suppose that $n+1$ and $-n-1$ are in different pairs. Then \check{x}' can be written as

$$\check{x}' = \left\{ \{i_1, i_2\}, \dots, \{i_m, -n-1\}, \dots, \{i_k, n+1\}, \dots, \{i_{2n+1}, i_{2n+2}\} \right\}.$$

In this case \check{x} is obtained from \check{x}' by removing $-n-1, n+1$ from pairs $\{i_m, -n-1\}$ and $\{i_k, n+1\}$ correspondingly, and by replacing these two pairs, $\{i_m, -n-1\}$ and $\{i_k, n+1\}$, by one pair $\{i_m, i_k\}$. The map $\check{x}' \rightarrow \check{x}$, denoted by $p_{n,n+1}$, will be referred to as the canonical projection of $X(n+1)$ onto $X(n)$.

Consider the sequence

$$X(1) \leftarrow \dots \leftarrow X(n) \leftarrow X(n+1) \leftarrow \dots$$

of canonical projections, and let

$$X = \varprojlim X(n)$$

denote the projective limit of the sets $X(n)$. By definition, the elements of X are arbitrary sequences $\check{x} = (\check{x}_1, \check{x}_2, \dots)$, such that $\check{x}_n \in X(n)$, and $p_{n,n+1}(\check{x}_{n+1}) = \check{x}_n$. The set X is a closed subset of the compact space of all sequences (\check{x}_n) , therefore, it is a compact space itself.

In what follows we denote by p_n the projection $X \rightarrow X(n)$ defined by $p_n(\check{x}) = \check{x}_n$.

Let \check{x} be an element of $X(n)$. Then \check{x} can be identified with arrow configurations on circles. Such arrow configurations can be constructed as follows. Once \check{x} is written as a collection of n unordered pairs, one can represent \check{x} as a union of cycles of the form

$$(2.2) \quad j_1 \rightarrow -j_2 \rightarrow j_2 \rightarrow -j_3 \rightarrow j_3 \rightarrow \dots \rightarrow -j_k \rightarrow j_k \rightarrow -j_1 \rightarrow j_1,$$

where j_1, j_2, \dots, j_k are distinct integers from the set $\{-n, \dots, n\}$.

For example, take

$$(2.3) \quad \check{x} = \left\{ \{1, 3\}, \{-2, 5\}, \{2, -1\}, \{-3, -5\}, \{4, -6\}, \{-4, 6\} \right\}.$$

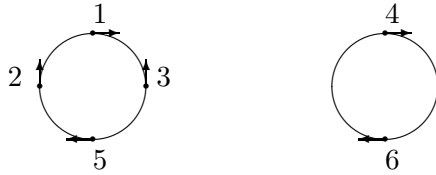


FIGURE 1. The representation of the element

$$\check{x} = \left\{ \{1, 3\}, \{-2, 5\}, \{2, -1\}, \{-3, -5\}, \{4, -6\}, \{-4, 6\} \right\}$$

in terms of arrow configurations on circles. The first circle (from the left) represents cycle $1 \rightarrow 3 \rightarrow -3 \rightarrow -5 \rightarrow 5 \rightarrow -2 \rightarrow 2 \rightarrow -1 \rightarrow 1$, and the second circle represents cycle $4 \rightarrow -6 \rightarrow 6 \rightarrow -4 \rightarrow 4$.

Then $\check{x} \in X(3)$, and it is possible to think about \check{x} as a union of two cycles, namely

$$1 \rightarrow 3 \rightarrow -3 \rightarrow -5 \rightarrow 5 \rightarrow -2 \rightarrow 2 \rightarrow -1 \rightarrow 1,$$

and

$$4 \rightarrow -6 \rightarrow 6 \rightarrow -4 \rightarrow 4.$$

Cycle (2.2) can be represented as a circle with attached arrows. Namely, we put on a circle points labeled by $|j_1|, |j_2|, \dots, |j_k|$, and attach arrows to these points according to the following rules. The arrow attached to $|j_1|$ is directed clockwise. If the next integer in the cycle (2.2), j_2 , has the same sign as j_1 , then the direction of the arrow attached to $|j_2|$ is the same as the direction of the arrow attached to $|j_1|$, i.e. clockwise. Otherwise, if the sign of j_2 is opposite to the sign of j_1 , the direction of the arrow attached to $|j_2|$ is opposite to the direction of the arrow attached to $|j_1|$, i.e. counterclockwise. Next, if the integer j_3 has the same sign as j_2 , then the direction of the arrow attached to $|j_3|$ is the same as the direction of the arrow attached to $|j_2|$, etc. For example, the representation of the element \check{x} defined by (2.3) in terms of arrow configurations on circles is shown in Fig. 1.

2.2. The t -measures on X .

Definition 2.1. For $t > 0$ we set

$$\mu_t^{(n)}(\check{x}) = \frac{t^{[\check{x}]_n}}{t(t+2) \dots (t+2n-2)},$$

where $\check{x} \in X(n)$, and $[\check{x}]_n$ denotes the number of cycles in \check{x} , or the number of circles in the representation of \check{x} in terms of arrow configurations.

Remark 2.2. The measures $\mu_t^{(n)}$ on the spaces $X(n)$ are natural analogues of the Ewens measures on the group $S(n)$ described in Kerov, Olshanski and Vershik [14].

Proposition 2.3. *a) We have*

$$(2.4) \quad \sum_{\check{x} \in X(n)} \mu_t^{(n)}(\check{x}) = 1.$$

Thus $\mu_t^{(n)}(\check{x})$ can be understood as a probability measure on $X(n)$.

b) Given $t > 0$, the canonical projections $p_{n,n+1}$ preserve the measures $\mu_t^{(n)}(\check{x})$, which means that the condition

$$(2.5) \quad \mu_t^{(n+1)}\left(\{\check{x}' \mid \check{x}' \in X(n+1), p_{n,n+1}(\check{x}') = \check{x}\}\right) = \mu_t^{(n)}(\check{x})$$

is satisfied for each $\check{x} \in X(n)$.

Proof. See Strahov [22], Section 4.1 □

It follows from Proposition 2.3 that for any given $t > 0$, the canonical projection $p_{n-1,n}$ preserves the measures $\mu_t^{(n)}$. Hence the measure

$$\mu_t = \varprojlim \mu_t^{(n)}$$

on X is correctly defined, and it is a probability measure.

Now we describe the right action of the group $S(2n)$ on the space $X(n)$, and then we extend it to the right action of $S(2\infty)$ on X .

Let $\check{x}_n \in X(n)$. Then \check{x}_n can be written as a collection of n unordered pairs (equation (2.1)). Let g be a permutation from $S(2n)$,

$$g : \begin{pmatrix} -n & -n+1 & \dots & n-1 & n \\ g(-n) & g(-n+1) & \dots & g(n-1) & g(n) \end{pmatrix}.$$

The right action of the group $S(2n)$ on the space $X(n)$ is defined by

$$\check{x}_n \cdot g = \left\{ \{g(i_1), g(i_2)\}, \{g(i_3), g(i_4)\}, \dots, \{g(i_{2n-1}), g(i_{2n})\} \right\}.$$

Proposition 2.4. *The canonical projection $p_{n,n+1}$ is equivariant with respect to the right action of the group $S(2n)$ on the space $X(n)$, which means*

$$p_{n,n+1}(\check{x} \cdot g) = p_{n,n+1}(\check{x}) \cdot g,$$

for all $\check{x} \in X(n+1)$, and all $g \in S(2n)$.

Proof. See Strahov [22], Section 4.2 □

Since the canonical projection $p_{n,n+1}$ is equivariant, the right action of $S(2n)$ on $X(n)$ can be extended to the right action of $S(2\infty)$ on X . For $n = 1, 2, \dots$ we identify $S(2n)$ with the subgroup of permutations $g \in S(2n+2)$ preserving the elements $-n-1$ and $n+1$ of the set $\{-n-1, -n, \dots, -1, 1, \dots, n, n+1\}$, i.e.

$$S(2n) = \left\{ g \mid g \in S(2n+2), g(-n-1) = -n-1, \text{ and } g(n+1) = n+1 \right\}.$$

Let $S(2) \subset S(4) \subset S(6) \dots$ be the collection of such subgroups. Set

$$S(2\infty) = \bigcup_{n=1}^{\infty} S(2n).$$

Thus $S(2\infty)$ is the inductive limit of subgroups $S(2n)$,

$$S(2\infty) = \varinjlim S(2n).$$

If $\check{x} = (\check{x}_1, \check{x}_2, \dots) \in X$, and $g \in S(2\infty)$, then the right action of $S(2\infty)$ on $X = \varinjlim X(n)$,

$$X \times S(2\infty) \longrightarrow X,$$

is defined as $\check{x} \cdot g = \check{y}$, where $\check{x}_n \cdot g = \check{y}_n$ for all n so large that $g \in S(2\infty)$ lies in $S(2n)$.

Proposition 2.5. *We have*

$$p_n(\check{x} \cdot g) = p_n(\check{x}) \cdot g$$

for all $\check{x} \in X$, $g \in S(2\infty)$, and for all n so large that $g \in S(2n)$.

Proof. The claim follows immediately from the very definition of the projection p_n , and of the right action of $S(2\infty)$ on X . \square

Proposition 2.6. *For any $\check{x} = (\check{x}_n) \in X$, and $g \in S(2\infty)$, the quantity*

$$c(\check{x}; g) = [p_n(\check{x} \cdot g)]_n - [p_n(\check{x})]_n = [p_n(\check{x}) \cdot g]_n - [p_n(\check{x})]_n$$

does not depend on n provided that n is so large that $g \in S(2n)$.

Proof. See Strahov [22], Section 4.3. \square

Proposition 2.7. *Each of measures μ_t , $0 < t < \infty$, is quasiinvariant with respect to the action of $S(2\infty)$ on the space $X = \varinjlim X(n)$. More precisely,*

$$\frac{\mu_t(d\check{x} \cdot g)}{\mu_t(d\check{x})} = t^{c(\check{x}; g)}; \quad \check{x} \in X, \quad g \in S(2\infty),$$

where $c(\check{x}; g)$ is the fundamental cocycle.

Proof. See Strahov [22], Section 4.4 \square

2.3. Definition of $T_{z, \frac{1}{2}}$. Let $(\mathfrak{X}, \Sigma, \mu)$ be a measurable space. Let G be a group which acts on \mathfrak{X} from the right, and preserves the Borel structure. Assume that the measure μ is quasiinvariant, i.e. the condition

$$d\mu(\check{x} \cdot g) = \delta(\check{x}; g) d\mu(\check{x})$$

is satisfied for some nonnegative μ -integrable function $\delta(\check{x}; g)$ on \mathfrak{X} , and for every g , $g \in G$. Set

$$(2.6) \quad (T(g)f)(\check{x}) = \tau(\check{x}; g)f(\check{x} \cdot g), \quad f \in L^2(\mathfrak{X}, \mu),$$

where $|\tau(\check{x}; g)|^2 = \delta(\check{x}; g)$. If

$$\tau(\check{x}; g_1 g_2) = \tau(\check{x} \cdot g_1; g_2) \tau(\check{x}; g_1), \quad \check{x} \in \mathfrak{X}, g_1, g_2 \in G,$$

then equation (2.6) defines a unitary representation T of G acting in the Hilbert space $L^2(\mathfrak{X}; \mu)$. The function $\tau(\check{x}; g)$ is called a multiplicative cocycle.

Let $z \in \mathbb{C}$ be a nonzero complex number. We apply the general construction described above for the space $\mathfrak{X} = X$, the group $G = S(2\infty)$, the measure $\mu = \mu_t$ (where $t = |z|^2$), and the cocycle $\tau(\check{x}; g) = z^{c(\check{x}; g)}$. In this way we get a unitary

representation of $S(2\infty)$, $T_{z, \frac{1}{2}}$, acting in the Hilbert space $L^2(X, \mu_t)$ according to the formula

$$\left(T_{z, \frac{1}{2}}(g)f\right)(\check{x}) = z^{c(\check{x};g)} f(\check{x} \cdot g), \quad f \in L^2(X, \mu_t), \quad \check{x} \in X, \quad g \in S(2\infty).$$

This defines a family of unitary representations of $S(2\infty)$.

2.4. Spherical functions. Let (G, K) be a Gelfand pair, and let T be a unitary representation of G acting in the Hilbert space $H(T)$. Assume that ξ is a unit vector in $H(T)$ such that ξ is K -invariant, and such that the span of vectors of the form $T(g)\xi$ (where $g \in G$) is dense in $H(T)$. In this case ξ is called the spherical vector, and the matrix coefficient $(T(g)\xi, \xi)$ is called the spherical function of the representation T . Two spherical representations are equivalent if and only if their spherical functions coincide.

The representation $T_{z, \frac{1}{2}}$ is realized in the Hilbert space $L^2(X, \mu_t)$, where $t = |z|^2$. Let $\mathbf{1}$ denote the function on X identically equal to 1. It can be viewed as an element of $L^2(X, \mu_t)$, and as a spherical vector of $T_{z, \frac{1}{2}}$. Set

$$\varphi_z(g) = \left(T_{z, \frac{1}{2}}(g)\mathbf{1}, \mathbf{1}\right), \quad g \in S(2\infty).$$

Thus φ_z is the spherical function of $T_{z, \frac{1}{2}}$.

3. SPECTRAL REPRESENTATION OF THE SPHERICAL FUNCTION OF $T_{z, \frac{1}{2}}$

3.1. The z -measures on partitions with the general parameter $\theta > 0$. Denote the set of Young diagrams with n boxes by \mathbb{Y}_n . Let $M_{z, \bar{z}, \theta}^{(n)}$ be a probability measure on \mathbb{Y}_n defined by

$$(3.1) \quad M_{z, \bar{z}, \theta}^{(n)} = \frac{n!(z)_{\lambda, \theta}(\bar{z})_{\lambda, \theta}}{\left(\frac{z\bar{z}}{\theta}\right)_n H(\lambda, \theta) H'(\lambda, \theta)},$$

where $n = 1, 2, \dots$, and where we use the following notation

- $z \in \mathbb{C}$ and $\theta > 0$ are parameters.
- $(a)_n$ stands for the Pochhammer symbol,

$$(a)_n = a(a+1) \dots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

- $(z)_{\lambda, \theta}$ is a multidimensional analogue of the Pochhammer symbol defined by

$$(z)_{\lambda, \theta} = \prod_{(i,j) \in \lambda} (z + (j-1) - (i-1)\theta) = \prod_{i=1}^{l(\lambda)} (z - (i-1)\theta)_{\lambda_i}.$$

Here $(i, j) \in \lambda$ stands for the box in the i th row and the j th column of the Young diagram λ , and we denote by $l(\lambda)$ the number of nonempty rows in the Young diagram λ .

•

$$H(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + 1),$$

$$H'(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + \theta),$$

where λ' denotes the transposed diagram.

Proposition 3.1. *We have*

$$M_{z, \bar{z}, \theta}^{(n)}(\lambda) = M_{-z/\theta, -\bar{z}/\theta, 1/\theta}^{(n)}(\lambda').$$

The probability measures $M_{z, \bar{z}, \theta}^{(n)}$ are called the z -measures with an arbitrary Jack parameter $\theta > 0$.

3.2. The spectral z -measures with the general parameter $\theta > 0$.

Definition 3.2. The space Ω of all pairs $\omega = (\alpha, \beta)$ of weakly decreasing sequences of non-negative real numbers,

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k \geq \dots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \dots \geq \beta_k \geq \dots \geq 0)$$

such that

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k \leq 1$$

is called the Thoma set.

Let us define an embedding of the algebra Λ of symmetric functions into the algebra of continuous functions on the Thoma set Ω . Since $\Lambda = \mathbb{C}[p_1, p_2, \dots]$, where $p_k = \sum_i x_i^k$ are power sums, it is sufficient to define the images \tilde{p}_k of the p_k 's. We set

$$(3.2) \quad \tilde{p}_k(\omega|\theta) = \begin{cases} \sum_{j=1}^{\infty} \alpha_j^k + (-\theta)^{k-1} \sum_{j=1}^{\infty} \beta_j^k, & k = 2, 3, \dots, \\ 1, & k = 1. \end{cases}$$

Then the following result holds true (see Borodin and Olshanski [7], Section 1):

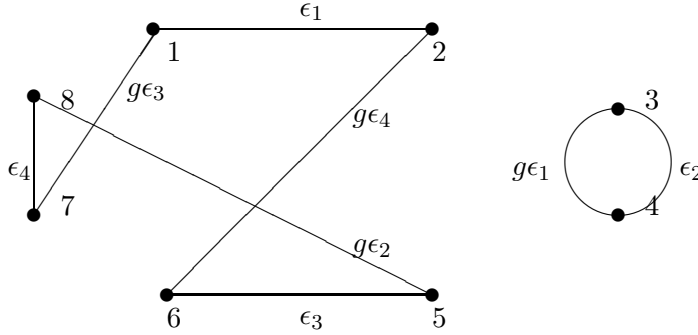
Theorem 3.3. *For any $n = 1, 2, \dots$ and any $\lambda \in \mathbb{Y}_n$ we have*

$$M_{z, \bar{z}, \theta}^{(n)}(\lambda) = \frac{n!}{H(\lambda, \theta)} \int_{\omega = (\alpha, \beta) \in \Omega} \tilde{P}_\lambda(\omega|\theta) M_{z, \bar{z}, \theta}^{\text{Spectral}}(d\omega),$$

where $\tilde{P}_\lambda(\omega|\theta)$ denotes the image of the Jack symmetric function $P_\lambda(x|\theta)$ (with the parameter θ) under the embedding defined by equation (3.2). Here $M_{z, \bar{z}, \theta}^{\text{Spectral}}$ is a unique probability measure on Ω .

Remark 3.4. 1) For the definition of the Jack symmetric functions $P_\lambda(x|\theta)$ see Macdonald [15], VI, §10. Note that our parameter θ is inverse to Macdonald's $\alpha = 1/\theta$.

2) The claim of Theorem 3.3 is a consequence of a more general statement proved in Kerov, Okounkov, Olshanski [12].


 FIGURE 2. Graph $\Gamma(g)$, where $g = (135)(67)(248) \in S(8)$.

3.3. A formula for the spherical function. Let φ_z be the spherical function of $T_{z, \frac{1}{2}}$. In what follows we describe the expansion of φ_z in terms of the zonal spherical functions of the Gelfand pair $(S(2n), H(n))$. The fact that $(S(2n), H(n))$ is a Gelfand pair implies that there is an orthogonal basis $\{w^\lambda\}$ in $C(S(2n), H(n))$ whose elements, w^λ , are the zonal spherical functions of $(S(2n), H(n))$. The elements w^λ are parameterized by Young diagrams with n boxes, and are defined by

$$w^\lambda(g) = \frac{1}{|H(n)|} \sum_{h \in H(n)} \chi^{2\lambda}(gh),$$

see Macdonald [15], Sections VII.1 and VII.2. Here $|H(n)|$ is the number of elements in the hyperoctahedral group of degree n , and $\chi^{2\lambda}$ is the character of the irreducible $S(2n)$ -module corresponding to $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$.

Proposition 3.5. *Denote by φ_z the spherical function of $T_{z, \frac{1}{2}}$. We have*

$$(3.3) \quad \varphi_z|_{S(2n)}(g) = \sum_{|\lambda|=n} M_{z, \bar{z}, \theta=\frac{1}{2}}^{(n)}(\lambda) w^\lambda(g), \quad g \in S(2n),$$

where $M_{z, \bar{z}, \theta=\frac{1}{2}}^{(n)}(\lambda)$ is the z -measure with the Jack parameter $\theta = 1/2$.

Proof. See Strahov [22], Proposition 6.3 □

3.4. Spectral representation. Recall the definition of the coset type of a permutation from $S(2n)$ (Macdonald [15], VII, §2). If $g \in S(2n)$, then we can associate with g a graph $\Gamma(g)$ with vertices $1, 2, \dots, 2n$, and edges $\epsilon_i, g\epsilon_i$ ($1 \leq i \leq n$). Each edge ϵ_i joins the vertices $2i - 1$ and $2i$, and $g\epsilon_i$ joins the vertices $g(2i - 1)$ and $g(2i)$. Then the connected components of $\Gamma(g)$ can be understood as cycles of even lengths $2\rho_1, 2\rho_2, \dots$, where $\rho_1 \geq \rho_2 \geq \dots$. Thus each $g \in S(2n)$ gives rise to a partition $\rho = (\rho_1, \rho_2, \dots)$ of n , called the coset type of g . For example, if

$$g = (135)(67)(248) \in S(8),$$

then the graph $\Gamma(g)$ is shown on Fig.2, and the coset type of g is $\rho = (3, 1)$.

Theorem 3.6. *Let φ_z be the spherical function of the representation $T_{z, \frac{1}{2}}$. There exists a unique probability measure $M_{z, \theta = \frac{1}{2}}^{\text{Spectral}}$ on Ω such that for any $g \in S(2\infty)$*

$$\varphi_z(g) = \int_{\Omega} \chi_{\theta = \frac{1}{2}}^{(w)}(g) M_{z, \theta = \frac{1}{2}}^{\text{Spectral}}(d\omega),$$

where

$$(3.4) \quad \chi_{\theta = \frac{1}{2}}^{(w)}(g) = \prod_{k=2}^{\infty} (\tilde{p}_k(w|\theta = \frac{1}{2}))^{\rho_k(g)},$$

and where $\rho(g) = (\rho_1(g), \rho_2(g), \rho_3(g), \dots)$ is the coset type of $g \in S(2\infty)$.

Remark 3.7. a) The notion of coset-type for a permutation $g \in S(2n)$ can be extended to elements of $S(2\infty)$ in an obvious way.

b) The formula in Theorem 3.6 can be understood as a natural analogue of the spectral decomposition of the characters of the generalized regular representation of the infinite symmetric group, see Section 9.7 in Kerov, Olshanski, and Vershik [13, 14]. The functions $\chi_{\theta = \frac{1}{2}}^{(w)}(g)$ introduced in the statement of Theorem 3.6 play the role of the extreme characters of $S(\infty)$. Moreover, formula (3.4) is an analogue of an explicit formula for the extreme characters of the infinite symmetric group given by the Thoma theorem.¹

Proof. We use Theorem 3.3, Proposition 3.5, and find

$$\varphi_z|_{S(2n)}(g) = \int_{\Omega} \left(\sum_{|\lambda|=n} \frac{n!}{H(\lambda, \theta = \frac{1}{2})} \tilde{P}_{\lambda}(\omega|\theta = \frac{1}{2}) w^{\lambda}(g) \right) M_{z, \theta = \frac{1}{2}}^{\text{Spectral}}(d\omega)$$

Note that the zonal spherical functions w^{λ} of the Gelfand pair $(S(2n), H(n))$ are constant on the double cosets $H(n)gH(n)$ of $H(n)$ in $S(2n)$. Two permutations $g, g_1 \in S(2n)$ have the same coset-type if and only if $g_1 \in H(n)gH(n)$. Therefore for any $g \in S(2n)$ we can write

$$w^{\lambda}(g) = w_{\rho(g)}^{\lambda}, \quad \rho(g) \text{ is the coset type of } g \in S(2n).$$

Let $\tilde{J}_{\lambda}(\omega|\theta = \frac{1}{2})$ be the image of the Jack symmetric function J_{λ}^{α} with the parameter $\alpha = 2$ under the embedding defined by equation (3.2). The functions $\tilde{P}_{\lambda}(\omega|\theta = \frac{1}{2})$ are expressible in terms of $\tilde{J}_{\lambda}(\omega|\theta = \frac{1}{2})$, see Macdonald [15], VI, equation (10.22). Taking this into account it is not hard to see that the sum inside the integral above can be rewritten as

$$\sum_{|\lambda|=n} \frac{|H(n)|}{h(2\lambda)} w_{\rho(g)}^{\lambda} \tilde{J}_{\lambda}(\omega|\theta = \frac{1}{2}),$$

¹For a modern presentation of the Thoma theorem see Kerov, Olshanski, and Vershik [14], Section 9.

where $h(2\lambda)$ is the product of the hook-lengths of the partition 2λ . Formula (2.16) of Macdonald [15], VII, §2 gives

$$\sum_{|\lambda|=n} \frac{|H(n)|}{h(2\lambda)} w_{\rho(g)}^\lambda \tilde{J}_\lambda(\omega|\theta = \frac{1}{2}) = \prod_{k=2}^{\infty} \left(\tilde{p}_k(w|\theta = \frac{1}{2}) \right)^{\rho_k(g)},$$

and the statement of the Theorem follows. \square

4. THE POINT PROCESS DEFINED BY $M_{z,\theta=\frac{1}{2}}^{\text{Spectral}}$

Now our aim is to describe the probability measures $M_{z,\theta=\frac{1}{2}}^{\text{Spectral}}$. We use the idea proposed by Borodin and Olshanski to view the infinite collection of parameters $(\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots)$ as random points distributed according to $M_{z,\theta=\frac{1}{2}}^{\text{Spectral}}$. In this way we obtain a point process, and the correlation functions of this point process provide a detailed description of $M_{z,\theta=\frac{1}{2}}^{\text{Spectral}}$.

It is more convenient to work with lifted spectral measures $\tilde{M}_{z,\theta=\frac{1}{2}}^{\text{Spectral}}$ defined as follows. Denote by $\tilde{\Omega}$ the set of triples

$$\tilde{\omega} = (\alpha, \beta, \delta) \in \mathbb{R}^{2\infty} \times \mathbb{R}_{\geq 0},$$

where

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0), \beta = (\beta_1 \geq \beta_2 \geq \dots \geq 0), \delta \in \mathbb{R}_{\geq 0},$$

and

$$\sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq \delta.$$

Consider the map

$$((\alpha, \beta), \delta) \in \Omega \times \mathbb{R}_{\geq 0} \rightarrow (\delta\alpha, \delta\beta, \delta) \in \tilde{\Omega}.$$

By definition, the measure $\tilde{M}_{z,\theta=\frac{1}{2}}^{\text{Spectral}}$ is the pushforward of the measure

$$M_{z,\theta=\frac{1}{2}}^{\text{Spectral}} \otimes \left(\frac{s^{2z\bar{z}-1}}{\Gamma(2z\bar{z})} e^{-s} ds \right)$$

on $\Omega \times \mathbb{R}_{\geq 0}$ under this map. The procedure to obtain the probability measure $\tilde{M}_{z,\theta=\frac{1}{2}}^{\text{Spectral}}$ from the probability measure $M_{z,\theta=\frac{1}{2}}^{\text{Spectral}}$ is called lifting.² We define the embedding $\tilde{\Omega} \rightarrow \text{Conf}(\mathbb{R} \setminus \{0\})$ as

$$\tilde{\omega} = (\alpha, \beta, \delta) \rightarrow C = \{\delta\alpha_i \neq 0\} \cup \{-\delta\beta_j \neq 0\}.$$

This way we convert $\tilde{\omega}$ to a point configuration C in $\mathbb{R} \setminus \{0\}$. Given a probability measure on $\tilde{\Omega}$, its pushforward under this embedding is a probability measure on point configurations in $\mathbb{R} \setminus \{0\}$, i.e. a point process. In particular, the probability measure $\tilde{M}_{z,\theta=\frac{1}{2}}^{\text{Spectral}}$ defines a point process in $\mathbb{R} \setminus \{0\}$.

²This procedure was first introduced in Borodin [2] in the context of determinantal processes relevant for the harmonic analysis on the infinite symmetric group. It leads to an essential simplification of the correlation functions.

Let $F : \mathbb{R}_{>0}^n \rightarrow \mathbb{C}$ be any continuous function with compact support. The equality

$$\int_{w=(\alpha,\beta) \in \Omega} \sum_{i_1, \dots, i_k} F(\alpha_{i_1}, \dots, \alpha_{i_n}) \widetilde{M}_{z, \theta=\frac{1}{2}}^{\text{Spectral}}(dw) = \int_{\mathbb{R}^n > 0} F(x_1, \dots, x_n) \varrho_n^{\text{Spectral}}(dx)$$

(where the sum is taken over pairwise distinct indexes) defines the correlation measures $\varrho_n^{\text{Spectral}}(dx)$. The correlation functions $\varrho_n^{\text{Spectral}}(x)$ are densities of $\varrho_n^{\text{Spectral}}(dx)$ with respect to the Lebesgue measure dx . These correlation functions, $\varrho_n^{\text{Spectral}}(x)$, describe correlations of $\{\alpha_i\}$. It is possible to consider joint correlations of $\{\alpha_i\}$ and $\{\beta_i\}$, but in this paper we deal with correlations of $\{\alpha_i\}$ only.

In order to present the main result of the paper let us introduce the following notation. First, we introduce the functions $w_a(x; z, z')$ indexed by $a \in \mathbb{Z} + \frac{1}{2}$, parameterized by two complex parameters z and z' , and whose argument x varies in $\mathbb{R}_{>0}$. They are expressed through the classical Whittaker functions $W_{k,m}(x)$, see Andrews, Askey, and Roy [1], Section 4, for a definition. The functions $w_a(x; z, z')$ are defined in terms of $W_{k,m}(x)$ as

$$(4.1) \quad w_a(x; z, z') = \left(\Gamma(z - a + \frac{1}{2}) \Gamma(z' - a + \frac{1}{2}) \right)^{-\frac{1}{2}} x^{-\frac{1}{2}} W_{\frac{z+z'}{2}-a, \frac{z-z'}{2}}(x).$$

Since $W_{k,\mu}(x) = W_{k,-\mu}(x)$, this expression is symmetric with respect to $z \longleftrightarrow z'$.

Second, we introduce a function of two arguments called the (scalar) Whittaker kernel. This function is parameterized by two complex parameters, z and z' , and can be written as

$$(4.2) \quad K_{z,z'}^W(x, y) = \sqrt{zz'} \frac{w_{-\frac{1}{2}}(x; z, z') w_{\frac{1}{2}}(y; z, z') - w_{\frac{1}{2}}(x; z, z') w_{-\frac{1}{2}}(y; z, z')}{x - y}.$$

Theorem 4.1. *For any $z \in \mathbb{C} \setminus \{0\}$*

$$(4.3) \quad \varrho_n^{\text{Spectral}}(x_1, \dots, x_n) = \text{Pf} \left[\mathbb{K}^{\text{Spectral}}(x_i, x_j) \right]_{i,j=1}^n.$$

The 2×2 matrix valued correlation kernel $\mathbb{K}^{\text{Spectral}}(x, y)$ can be written as

$$\mathbb{K}^{\text{Spectral}}(x, y) = \begin{bmatrix} S(x, y) & \frac{d}{dy} S(x, y) \\ \frac{d}{dx} S(x, y) & \frac{d^2}{dx dy} S(x, y) \end{bmatrix},$$

where

$$S(x, y) = -\frac{1}{2} \sqrt{y} \int_x^{+\infty} K_{-2z, -2\bar{z}}^W(s, y) \frac{ds}{\sqrt{s}} \\ + \frac{\sqrt{z\bar{z}}}{4} \left(\int_x^{+\infty} w_{-\frac{1}{2}}(s; -2z, -2\bar{z}) \frac{ds}{\sqrt{s}} \right) \left(\int_y^{+\infty} w_{\frac{1}{2}}(t; -2z, -2\bar{z}) \frac{dt}{\sqrt{t}} \right).$$

Remark 4.2. a) All matrix elements of $\mathbb{K}^{\text{Spectral}}(x, y)$ are constructed in terms of the Whittaker functions.

b) From the definition of the Whittaker functions it follows that $\mathbb{K}^{\text{Spectral}}(x, y)$ is real valued.

c) It can be shown that for any $z \in \mathbb{C} \setminus \{0\}$ the following condition is satisfied

$$S(x, y) = -S(y, x).$$

Thus the matrix inside the Pfaffian in equation (4.3) is antisymmetric.

d) The function $K_{z, \bar{z}}^W(x, y)$ is the (scalar) correlation kernel of a determinantal process which arises in the study of the decomposition of the generalized regular representation of $S(\infty)$ into irreducible components, see Borodin and Olshanski [8].

5. IDEA OF THE PROOF OF THEOREM 4.1

The first step in the proof of Theorem 4.1 is to construct a point process on the lattice $\mathbb{Z} + \frac{1}{2}$, which converges to the point process defined by $\widetilde{M}_{z, \theta=\frac{1}{2}}^{\text{Spectral}}$. Given a box $b = (i, j)$ of a Young diagram and a parameter $\theta > 0$ the number

$$c_\theta(b) = (j - 1) - \theta(i - 1)$$

is referred to as θ -content of the box b . A box b of a Young diagram is said to be positive or negative according to the sign of its θ -content. We can consider any Young diagram $\lambda = (\lambda_1, \dots, \lambda_l)$ as a collection of boxes

$$\lambda \equiv \{b(i, j) : 1 \leq i \leq l, 1 \leq j \leq \lambda_i\},$$

and we can split λ into a union of disjoint subsets of its positive and negative boxes

$$\lambda^+ = \{b \in \lambda | c_\theta(b) > 0\}, \quad \lambda^- = \{b \in \lambda | c_\theta(b) \leq 0\}.$$

Denote by r the number of rows in λ^+ , and by s the number of columns in λ^- . Let

$$a_1 \geq a_2 \geq \dots \geq a_r > 0, \quad b_1 \geq b_2 \geq \dots \geq b_s > 0$$

denote the lengths of corresponding rows and columns. Then the expression

$$\lambda = \left(a_1, \dots, a_r \middle| b_1, \dots, b_s \right)$$

defines coordinates of the Young diagram λ , and these coordinates are dependent on the Jack parameter θ . Now each Young diagram λ can be represented as a point configuration on $\mathbb{Z} + \frac{1}{2}$ using the map

$$\lambda \longrightarrow (A|B)_\theta(\lambda) = \left(-a_1 - \frac{1}{2}, \dots, -a_r - \frac{1}{2}; b_1 + \frac{1}{2}, \dots, b_s + \frac{1}{2} \right).$$

Next define a probability measure $\widetilde{M}_{z, \bar{z}, \theta, \xi}$ on the set of all Young diagrams \mathbb{Y} by

$$(5.1) \quad \widetilde{M}_{z, \bar{z}, \theta, \xi} = \left(1 - \frac{z\bar{z}}{\theta} \right)^{\frac{z\bar{z}}{\theta}} \frac{\left(\frac{z\bar{z}}{\theta} \right)^n}{n!} \xi^n M_{z, \bar{z}, \theta}^{(n)}(\lambda), \quad |\lambda| = n.$$

Here $0 < \xi < 1$ is an additional parameter. We refer to $\widetilde{M}_{z, \bar{z}, \theta, \xi}$ as to the mixed z -measures with an arbitrary Jack parameter $\theta > 0$. Define the lattice correlation functions $\varrho_{n, \theta, \xi}$ with the general parameter $\theta > 0$ as probabilities that the set $(A|B)_\theta(\lambda)$ contains a fixed subset X of $\mathbb{Z}_{\geq 0} + \frac{1}{2}$. More precisely,

$$\varrho_{n, \theta, \xi}(X) = \widetilde{M}_{z, \bar{z}, \theta, \xi}(\lambda | X \subset (A|B)_\theta(\lambda)), \quad X \subset \mathbb{Z}_{\geq 0} + \frac{1}{2}.$$

The crucial fact which enables us to compute the correlation functions of $\widetilde{M}_{z, \theta=\frac{1}{2}}^{\text{Spectral}}$, and to prove Theorem 4.1 is the following

Proposition 5.1. *Let $\xi \nearrow 1$, and assume that $x_1, \dots, x_n \rightarrow +\infty$ inside $\mathbb{Z}_{\geq 0} + \frac{1}{2}$ such that*

$$(1 - \xi)x_1 \rightarrow u_1, \dots, (1 - \xi)x_n \rightarrow u_n,$$

where u_1, \dots, u_n are pairwise distinct points of $\mathbb{R}_{>0}$.

Then we have

$$\varrho_n^{\text{Spectral}}(u_1, \dots, u_n) = \lim_{\xi \nearrow 1} (1 - \xi)^{-n} \varrho_{n, \theta=\frac{1}{2}, \xi}(x_1, \dots, x_n).$$

(This statement can be proved by repetition of arguments from Borodin and Olshanski [7]). Proposition 5.1 shows that in order to compute the correlation function ϱ_n , it is necessary to have a formula for the correlation function $\varrho_{n, \theta=\frac{1}{2}, \xi}$ defined by the mixed z -measure with the Jack parameter $\theta = \frac{1}{2}$. By Proposition 3.1 it is enough to compute the correlation functions $\varrho_{n, \theta=2, \xi}$ defined by the mixed z -measure with the Jack parameter parameter $\theta = 2$. Such correlation functions were given in Strahov [23]. It was shown (see Strahov [23], Section 2) that the correlation functions $\varrho_{n, \theta=2, \xi}$ can be written as Pfaffians with 2×2 matrix valued kernels. Moreover, these kernels are expressible in terms of the Gauss hypergeometric functions, and for matrix elements of these kernels there are double contour integral representations (see Strahov [23], Proposition 2.9, Proposition 2.10, equation (2.6)). Using these results it is possible to compute the scaling limit of the correlation functions $\varrho_{n, \theta=2, \xi}$ as $\xi \nearrow 1$, and to obtain the formula for $\varrho_n^{\text{Spectral}}$ in Theorem 4.1.

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